

Von Neumann entropy and majorization [‡]

Yuan Li^a, Paul Busch^b

*a. College of Mathematics and Information Science, Shaanxi Normal University,
Xi'an, 710062, People's Republic of China.*

b. Department of Mathematics, University of York, York, YO10 5DD, United Kingdom

Abstract: In this paper, we firstly consider the properties of the Shannon entropy for two probability distributions which stand in the relationship of majorization. Then we give a generalized Uhlmann theorem in an infinite dimension Hilbert space. Also, we show that $S(\Phi(\rho)) = S(\rho)$ for all quantum states ρ if and only if there exists an isometry operator V such that $\Phi(\rho) = V\rho V^*$, where Φ is a quantum channel.

Keywords: Von Neumann entropy, majorization, Quantum operation

AMS Classification: 47L05, 47L90, 81R10

1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the von Neumann algebra of all bounded linear operators on a separable Hilbert space \mathcal{H} and $S(\mathcal{H})$ be the set of all density operators on \mathcal{H} . That is $\rho \in S(\mathcal{H})$ if and only if $\rho \geq 0$ and $\text{tr}(\rho) = 1$. Each element of $S(\mathcal{H})$ is also called a quantum state in quantum information. Let $\mathcal{T}(\mathcal{H})$ be the set of all trace class operators on \mathcal{H} . An operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called an isometry if $V^*V = I_{\mathcal{H}}$. In this case, $VV^* \in \mathcal{B}(\mathcal{K})$ is an orthogonal projection. As usual, for $x, y \in \mathcal{H}$, $x \otimes y$ denotes the operator $z \rightarrow \langle z, y \rangle x$ ($z \in \mathcal{H}$). For $\rho \in S(\mathcal{H})$, we denote $\lambda(\rho) = (\lambda_1(\rho), \lambda_2(\rho), \dots, \lambda_n(\rho), \dots)$, which is the sequence of eigenvalues of ρ with the non-increasing order. Thus $\lambda(\rho) \in c_0^*$, where c_0^* is the positive cone of sequences decreasing monotonically to 0 as in [7]. Let $l^\infty(\mathbb{R})$ and $l_1^1(\mathbb{R}^+)$ denote the set of all bounded real sequences and all summable non-negative real sequences, which have sum 1, respectively. For a vector $r \in l^\infty(\mathbb{R})$, let $r^\downarrow = (r_1^\downarrow, \dots, r_n^\downarrow, \dots)$ denote the vector whose elements are the elements of r re-ordered into non-increasing order. Adopting the definition of majorization in [2, 7], for $r, s \in c_0^*$, we say that r is majorized by s , written as $r \prec s$, if

$$\sum_{i=1}^k r_i^\downarrow \leq \sum_{i=1}^k s_i^\downarrow, \text{ for } k = 1, 2, \dots \quad \text{and} \quad \sum_{i=1}^{\infty} r_i^\downarrow = \sum_{i=1}^{\infty} s_i^\downarrow.$$

*E-mail address: liyuan0401@yahoo.com.cn; paul.busch@york.ac.uk

[†]

Recently, many mathematicians have paid much attention to the infinite dimensional Schur-Horn theorem and infinite majorization. In [11], A. Neumann has given properties of infinite majorization in $l^\infty(\mathbb{R})$ and V. Kaftal and G. Weiss in [7] have obtained interesting results for infinite majorization in c_0^* . Arveson and Kadison by different methods presented some other characterizations in [3]. Some related results are the following: if $r, s \in l_1^1(\mathbb{R}^+)$, then

$$r \prec s \iff r = Qs, \text{ with } Q_{ij} = |U_{ij}|^2 \text{ for some unitary } U \text{ [6, Theorem 1]},$$

and

$$r \prec s \iff r = Qs, \text{ for some orthostochastic matrix } Q \text{ [7, Corollary 6.1]}.$$

Motivated by the above results, we firstly consider the properties of the Shannon entropy for two elements in $l_1^1(\mathbb{R}^+)$ which have majorization. Then we extend and study those properties for two operators $\rho, \sigma \in S(\mathcal{H})$. Following the definition given for finite dimensional spaces (see [1]), we denote $\rho \prec \sigma$ for two operators $\rho, \sigma \in S(\mathcal{H})$ if $\lambda(\rho) \prec \lambda(\sigma)$.

Let $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ be the von Neumann algebra of $n \times n$ matrices whose entries are in $\mathcal{B}(\mathcal{H})$ and $\Phi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be a linear map. Then Φ induces a map $\Phi_n : \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \longrightarrow \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ by the formula

$$\Phi_n((a_{i,j})) = (\Phi(a_{i,j})), \text{ for } (a_{i,j}) \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})).$$

If every Φ_n is a positive map, then Φ is called completely positive. Φ is said to be normal if Φ is continuous with respect to the ultraweak (σ -weak) topology. Normal completely positive contractive maps on $\mathcal{B}(\mathcal{H})$ were characterized in a theorem of Kraus [8, Theorem 3.3], which says that Φ is a normal completely positive map if and only if there exists a sequence $\{A_i\}_{i=1}^\infty$ of $\mathcal{B}(\mathcal{H})$ such that for all $X \in \mathcal{B}(\mathcal{H})$,

$$\Phi(X) = \sum_{i=1}^\infty A_i X A_i^* \quad \text{with} \quad \sum_{i=1}^\infty A_i A_i^* \leq I,$$

where the sequence $\{A_i\}_{i=1}^\infty$ is not necessarily unique and $\sum_{i=1}^\infty A_i A_i^* \leq I$ in the strong operator topology. The family $\{A_i\}_{i=1}^\infty$ is also called a family of Kraus operators for Φ . In this case, the dual of Φ is defined by

$$\Phi^\dagger(X) = \sum_{i=1}^\infty A_i^* X A_i \quad \text{for } X \in \mathcal{T}(\mathcal{H}).$$

It is easy to see that $|tr[\Phi^\dagger(X)Y]| = |tr[X\Phi(Y)]| \leq \|\Phi\| \|Y\| |tr(X)|$, for $X \in \mathcal{T}(\mathcal{H})$ and $Y \in \mathcal{B}(\mathcal{H})$, so $\Phi^\dagger(X) \in \mathcal{T}(\mathcal{H})$ is well defined on $\mathcal{T}(\mathcal{H})$. In general, Φ^\dagger cannot be an extension from $\mathcal{T}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. However, if $\sum_{i=1}^\infty A_i^* A_i \leq I$, then Φ^\dagger is well defined on $\mathcal{B}(\mathcal{H})$ and Φ^\dagger is normal. A normal completely positive map Φ which is trace preserving ($\Phi^\dagger(I) = I$) is called a quantum channel. If a normal completely positive map satisfy $\Phi(I) \leq I$, then Φ is called a quantum operation ([8,9]). A quantum operation is unital if $\Phi(I) = I$, which is equivalent to $\sum_j A_j A_j^* = I$. A quantum operation is bi-stochastic if it is both trace-preserving and unital. In particular, Φ is said to be a mixed

unitary operation if $\Phi(X) = \sum_{i=1}^n t_i U_i X U_i^*$, where $n < \infty$, U_i are all unitary operators and $t_i > 0$, $\sum_{i=1}^n t_i = 1$.

The von Neumann entropy of a quantum state ρ is defined by the formula

$$S(\rho) \equiv -\text{tr}(\rho \log(\rho))$$

As usual, in this formula logarithms are taken to base two. In classical information theory, the Shannon entropy is defined by $H(p) = -\sum_i p_i \log(p_i)$, where $p = (p_1, p_2 \cdots p_n \cdots)$ is a probability distribution. If λ_i are the eigenvalues of ρ , then the von Neumann entropy can be re-expressed as

$$S(\rho) = -\sum_{i=1}^{\infty} \lambda_i \log(\lambda_i) = H(\lambda(\rho)),$$

where we use $0 \log 0 = 0$.

In [4], Hardy, Littlewood and Pólya showed for $\xi, \eta \in \mathbb{R}^n$,

$$\xi \prec \eta \iff \xi = Q\eta \text{ for some doubly stochastic matrix } Q.$$

In the quantum context, Uhlmann obtained

$$\rho \prec \sigma \iff \rho = \Phi(\sigma), \Phi \text{ is a mixed unitary quantum operation,}$$

for $\rho, \sigma \in S(\mathcal{H})$, where \mathcal{H} is a finite dimension Hilbert space. This Uhlmann theorem might be used to study the connection between majorization and quantum mechanics. Furthermore, the quantum operations which can preserve the von Neumann entropy and the relative entropy of quantum states were extensively studied in more recent papers [5,10,15,17].

In this paper, we firstly consider the properties of Shannon entropy for two probability distributions which have majorization. Then we give a generalization of the result due to Uhlmann [1] in an infinite dimensional Hilbert space. Also, we show that $S(\Phi(\rho)) = S(\rho)$ for all quantum states ρ if and only if there exists an isometric operator V such that $\Phi(\rho) = V\rho V^*$, where Φ is a quantum channel.

2 Shannon entropy of infinite probability distributions

The following lemma is a direct corollary of [13, Theorem 8.0.1] as the function $f(x) := -x \log(x)$ is a concave function.

Lemma 2.1. Let $a, b \in \mathbb{R}^n$ with $a_i, b_i \geq 0$, and $a \prec b$. Then $\sum_{i=1}^n f(a_i) \geq \sum_{i=1}^n f(b_i)$.

The following is an extension of above lemma from the finite case to the infinite majorization.

Theorem 2.2. Let $a, b \in l_1^1(\mathbb{R}^+)$ and $a \prec b$. Then $H(a) \geq H(b)$.

Proof. Suppose that $c = (c_1, c_2, \dots, c_n, \dots)$, where $c_i \geq 0$ and $\sum_{i=1}^{\infty} c_i \leq 1$. It is clear that for arbitrary n , we have

$$(c_1, c_2, \dots, c_n, 0, 0, 0, \dots) \prec (\sum_{i=1}^n c_i, 0, 0, 0, \dots), \quad (2.1)$$

so Lemma 2.1 implies $-\sum_{i=1}^n c_i \log c_i \geq -(\sum_{i=1}^n c_i) \log(\sum_{j=1}^n c_j)$. Letting $n \rightarrow \infty$, by the continuity of the function $f(x) = -x \log x$, we get

$$-\sum_{i=1}^{\infty} c_i \log c_i \geq -\sum_{i=1}^{\infty} c_i \log(\sum_{j=1}^{\infty} c_j). \quad (2.2)$$

Firstly, we assume that b has only finite nonzero elements. Without loss of generality, we suppose

$$(a_1, a_2, \dots, a_n, \dots) = a \prec b = (b_1, b_2, \dots, b_m, 0, 0, \dots), \quad (2.3)$$

where $b_i > 0$, for $1 \leq i \leq m$. Thus there exists N such that $\sum_{i=N+1}^{\infty} a_i < \min\{b_1, b_2, \dots, b_m\}$, so

$$(a_1, a_2, \dots, a_N, \sum_{i=N+1}^{\infty} a_i, 0, 0, \dots) \prec (b_1, b_2, \dots, b_m, 0, 0, \dots), \quad (2.4)$$

which implies

$$-\sum_{i=1}^N a_i \log a_i - \sum_{i=N+1}^{\infty} a_i \log(\sum_{j=N+1}^{\infty} a_j) \geq -\sum_{i=1}^m b_i \log(b_i). \quad (2.5)$$

By inequality (2.2), we know $H(a) \geq H(b)$.

For the general case of b , we note that for any positive integers s , the relation $a \prec b$ entails

$$(a_1, a_2, \dots, a_n, \dots) \prec (b_1, b_2, \dots, b_s, \sum_{i=s+1}^{\infty} b_i, 0, 0, \dots), \quad (2.6)$$

so the proof above yields

$$H(a) \geq -\sum_{i=1}^s b_i \log b_i - \sum_{i=s+1}^{\infty} b_i \log(\sum_{i=s+1}^{\infty} b_i). \quad (2.7)$$

Letting $s \rightarrow \infty$, we get $H(a) \geq H(b)$, as desired.

Lemma 2.3. Let $a, b \in \mathbb{R}^n$ with $a_i, b_i \geq 0$. If $a \prec b$ and $\sum_{i=1}^n f(a_i) = \sum_{i=1}^n f(b_i)$ where $f(x) = -x \log(x)$, then $a^\downarrow = b^\downarrow$.

Proof. Without loss of generality, we assume that

$$a^\downarrow = (a_1, a_2, \dots, a_m, 0, \dots, 0) \prec b^\downarrow = (b_1, b_2, \dots, b_m, 0, \dots, 0), \quad (2.8)$$

where $m \leq n$ and $a_i + b_i \neq 0$, for $1 \leq i \leq m$. For $0 < t < 1$, denote

$$c_t = (ta_1 + (1-t)b_1, ta_2 + (1-t)b_2, \dots, ta_m + (1-t)b_m).$$

It is easy to verify that $a \prec c_t \prec b$, so Lemma 2.1 implies

$$-\sum_{i=1}^m (ta_i + (1-t)b_i) \log(ta_i + (1-t)b_i) = H(a). \quad (2.9)$$

Taking the second derivation function at t of the both sides of the equation(2.9), we get

$$\sum_{i=1}^m \frac{(a_i - b_i)^2}{ta_i + (1-t)b_i} = 0,$$

which yields $a_i = b_i$, for $1 \leq i \leq m$, so $a^\downarrow = b^\downarrow$.

The following proposition shows that the Shannon entropy of the infinite probability distribution is strictly monotone in the relation of majorization.

Proposition 2.4. Let $a, b \in l_1^1(\mathbb{R}^+)$ such that all $a_i b_i > 0$. If $a \prec b$ and $H(a) = H(b) < \infty$, then $a^\downarrow = b^\downarrow$.

Proof. Let

$$a^\downarrow = (a_1, a_2, \dots, a_n, \dots) \prec b^\downarrow = (b_1, b_2, \dots, b_n, \dots).$$

Then $a_1 \leq b_1$, so there exists k such that $a_1 \in (b_{k+1}, b_k]$, which implies $a_1 = tb_{k+1} + (1-t)b_k$, for some $0 \leq t \leq 1$. Denote

$$c = (b_1, b_2, \dots, tb_{k+1} + (1-t)b_k, (1-t)b_{k+1} + tb_k, b_{k+2}, \dots),$$

it is clear that $a \prec c \prec b$. Thus by Theorem 2.2 and the assumption, we have $H(a) = H(b) = H(c)$, so Lemma 2.1 implies $a_1 = tb_{k+1} + (1-t)b_k = b_k$. Let

$$\tilde{a} = (a_2, a_3, \dots, a_n, \dots) \text{ and } \tilde{b} = (b_1, b_2, \dots, b_{k-1}, b_{k+1}, \dots).$$

Thus $\tilde{a} \prec \tilde{b}$ and $H(\tilde{a}) = H(\tilde{b}) < \infty$, so by a similar proof, we conclude that there exists $k_1 > k$ such that $a_2 = b_{k_1}$. By mathematical induction, we get $a_{n+1} = b_{k_n}$, for some subsequence of b . However $\sum_{i=1}^\infty a_i = \sum_{i=1}^\infty b_i = 1$ and all $a_i b_i > 0$ imply $a_{n+1} = b_{k_n} = b_{n+1}$, that is $a^\downarrow = b^\downarrow$.

Remark 2.5. The conditions of all $a_i b_i > 0$ and $H(a) < \infty$ are essential in Proposition 2.4. Indeed, it is obvious that both conditions $a \prec b$ and $H(a) = H(b) < \infty$ are not changed if we add some zeros for a . Furthermore, if $H(a) = H(b) = \infty$, we may replace a by $a' = (\frac{a_1}{2}, \frac{a_1}{2}, a_2, \dots)$. Then $a' \prec b$ and $H(a') = H(b) = \infty$. However, it is a contradiction that $a^\downarrow = b^\downarrow = a'^\downarrow$.

3 von Neumann entropy of quantum states

Let us recall that two quantum states ρ_1 and ρ_2 are \mathcal{L}^1 -equivalent if and only if there is a sequence of unitary operators $\{U_i\}_{i=1}^\infty$ such that $\lim_{n \rightarrow \infty} \|\rho_1 - U_n \rho_2 U_n^*\|_1 = 0$.

Proposition 3.1. Let $\rho_1, \rho_2 \in S(\mathcal{H})$. If $\rho_1 \prec \rho_2$ and $S(\rho_1) = S(\rho_2) < \infty$, then ρ_1 and ρ_2 are \mathcal{L}^1 -equivalent.

Proof. By Proposition 2.4, all the non-zero spectral points (including the multiplicity) of ρ_1 and ρ_2 are the same, which is equivalent to the fact that ρ_1 and ρ_2 are \mathcal{L}^1 -equivalent, by [3, Proposition 3.1].

The following two results give extensions of Uhlmann theorem in an infinite dimensional Hilbert space. We need some eigenvalue estimates of Weyl given in [16]. Let A be a compact operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ and \mathcal{P}_n be the set of all n -dimensional projections. Then $\max_{P \in \mathcal{P}_n} \text{tr}(AP) = \sum_{i=1}^n \lambda_i$.

Proposition 3.2. Let $\rho_1, \rho_2 \in S(\mathcal{H})$. If ρ_1 is finite rank, then $\rho_1 \prec \rho_2$ if and only if there exists a mixed unitary operation Φ such that $\Phi(\rho_2) = \rho_1$.

Proof. Necessity. If ρ_1 is finite rank, then denote $\lambda(\rho_1) = (\lambda_1(\rho_1), \lambda_2(\rho_1) \cdots \lambda_m(\rho_1), 0, 0 \cdots)$. As $\rho_1 \prec \rho_2$, we have that ρ_2 is also finite rank and $\lambda(\rho_2) = (\lambda_1(\rho_2), \lambda_2(\rho_2) \cdots \lambda_n(\rho_2), 0, 0 \cdots)$ where $n \leq m$. By the spectral decomposition of states ρ_1 and ρ_2 , we conclude that there exist two orthonormal bases $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of \mathcal{H} such that

$$\rho_1 = \sum_{i=1}^m \lambda_i(\rho_1) x_i \otimes x_i \text{ and } \rho_2 = \sum_{i=1}^n \lambda_i(\rho_2) y_i \otimes y_i.$$

Let \mathcal{H}_1 be the subspace spanned by $\{x_i\}_{i=1}^m$ and U be the unitary operator defined by $Uy_i = x_i$, for $i = 1, 2, \dots$. Then $\rho_1|_{\mathcal{H}_1}$ and $U\rho_2U^*|_{\mathcal{H}_1}$ can be represented by $m \times m$ matrices, which have spectra $(\lambda_1(\rho_1), \lambda_2(\rho_1) \cdots \lambda_m(\rho_1))$ and $(\lambda_1(\rho_2), \lambda_2(\rho_2) \cdots \lambda_n(\rho_2), \dots, 0) \in \mathbb{R}^m$, respectively. Thus Uhlmann's theorem ([1 or 13 Theorem 4.1.1]) implies that there exists a mixed unitary operation $\tilde{\Phi}$ on $\mathcal{B}(\mathcal{H}_1)$, that is $\rho_1|_{\mathcal{H}_1} = \sum_{i=1}^m t_i V_i (U\rho_2U^*)|_{\mathcal{H}_1} V_i^*$, where $V_i \in \mathcal{B}(\mathcal{H}_1)$ are unitary operators, $\sum_{i=1}^m t_i = 1$ and all $t_i > 0$. As $(U\rho_2U^*)|_{\mathcal{H}_1^\perp} = 0$, we denote $U_i = \text{diag}(V_i, I_{\mathcal{H}_1^\perp})$, so $U_i \in \mathcal{B}(\mathcal{H})$ are unitary operators, and $\rho_1 = \sum_{i=1}^m t_i U_i (U\rho_2U^*) U_i^*$.

Sufficiency. By the Weyl estimates of [16], we get for $n = 1, 2, \dots$,

$$\begin{aligned} \sum_{i=1}^n \lambda_i(\rho_1) &= \max_{P \in \mathcal{P}_n} \{\text{Tr}(\rho_1 P)\} \\ &= \max_{P \in \mathcal{P}_n} \{\text{Tr}(\sum_{i=1}^m t_i U_i \rho_2 U_i^* P)\} \\ &= \max_{P \in \mathcal{P}_n} \{\sum_{i=1}^m t_i \text{Tr}(\rho_2 U_i^* P U_i)\} \\ &\leq \sum_{i=1}^m t_i \max_{P \in \mathcal{P}_n} \{\text{Tr}(\rho_2 U_i^* P U_i)\} \\ &= \max_{P \in \mathcal{P}_n} \{\text{Tr}(\rho_2 P)\} \\ &= \sum_{i=1}^n \lambda_i(\rho_2). \end{aligned}$$

Theorem 3.3. Let $\rho_1, \rho_2 \in S(\mathcal{H})$. Then the following three conditions are equivalent:

- (a) $\rho_1 \prec \rho_2$;
- (b) there exists a sequence of mixed unitary operation Ψ_n and a bi-stochastic quantum operation Ψ on $S(\mathcal{H})$ such that $\lim_{n \rightarrow \infty} \|\Psi_n(\rho) - \Psi(\rho)\|_1 = 0$ for all $\rho \in S(\mathcal{H})$, and $\Psi(\rho_2) = \rho_1$;
- (c) There exists a bi-stochastic quantum operation Ψ such that $\Psi(\rho_2) = \rho_1$.

Proof. (a) \implies (b). If $\rho_1 \prec \rho_2$, then $\lambda(\rho_1)^\downarrow \prec \lambda(\rho_2)^\downarrow$, it follows from [6, Theorem 1] that $\lambda(\rho_1)^\downarrow = Q\lambda(\rho_2)^\downarrow$, where Q is an infinite matrix satisfied $Q_{ij} = |U_{ij}|^2$ for a unitary operator U . By the spectral decomposition of states ρ_2 , there exists an orthonormal basis $\{y_i\}_{i=1}^\infty$ of \mathcal{H} such

that $\rho_2 = \sum_{i=1}^{\infty} \lambda_i(\rho_2) y_i \otimes y_i$. Let $\{x_i\}_{i=1}^{\infty}$ be the usual orthonormal basis of \mathcal{H} (in the sense of isomorphism), that is $x_i = \underbrace{(0, 0 \cdots 0, 1, 0 \cdots)}_i$. Denote $U_1 = \sum_{i=1}^{\infty} x_i \otimes y_i$ and $e_i = U_1^* U^* U_1 y_i$. Then it is easy to see that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of \mathcal{H} such that $\lambda_i(\rho_1) = \langle \rho_2 e_i, e_i \rangle$ for $i = 1, 2, \dots$. Thus ρ_2 has an infinite matrix form as (with respect to the basis $\{e_i\}_{i=1}^{\infty}$)

$$\rho_2 = \begin{pmatrix} \lambda_1(\rho_1) & \lambda_{12} & \cdots & \lambda_{1n} & \cdots \\ \lambda_{21} & \lambda_2(\rho_1) & \cdots & \lambda_{2n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n1} & \cdots & \cdots & \lambda_n(\rho_1) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3.10)$$

where $\lambda_{ij} = \overline{\lambda_{ji}}$. Denote the sequence of rank-one projections $E_i = e_i \otimes e_i$, then define $\Phi(\rho) = \sum_{i=1}^{\infty} E_i \rho E_i$, for $\rho \in S(\mathcal{H})$. Also for $n = 1, 2, \dots$, let Φ_n be the mixed unitary operation defined as follows $\Phi_n(\rho) = \frac{1}{n} \sum_{i=1}^n U^i \rho (U^i)^*$, where $U = \text{diag}(\omega, \omega^2 \cdots \omega^n, 1, 1 \cdots 1 \cdots)$ and $\omega = \exp \frac{2\pi\sqrt{-1}}{n}$. By a direct calculation, we get

$$\Phi_n(e_i \otimes e_j) = \frac{1}{n} \sum_{i=1}^n U^i e_i \otimes e_j (U^i)^* = \begin{cases} e_i \otimes e_i, & i = j \\ e_i \otimes e_j, & i, j > n \\ 0, & i \neq j \text{ and } i \leq n \text{ or } j \leq n. \end{cases}$$

Thus

$$\Phi_n(\rho_2) = \begin{pmatrix} \rho_{21} & 0 \\ 0 & \rho_{22} \end{pmatrix} : \left(\bigvee_{i=1}^n \{e_i\} \right) \oplus \left(\bigvee_{i=n+1}^{\infty} \{e_i\} \right), \quad (3.11)$$

where

$$\rho_{21} = \text{diag}(\lambda_1(\rho_1), \lambda_2(\rho_1) \cdots \lambda_n(\rho_1)), \quad \rho_{22} = (I - P_n) \rho_2 (I - P_n)|_{\left(\bigvee_{i=n+1}^{\infty} \{e_i\} \right)}, \quad (3.12)$$

and P_n denotes the orthogonal projection on the subspace $\bigvee_{i=1}^n \{e_i\}$. Then

$$\|\Phi_n(\rho_2) - \Phi(\rho_2)\|_1 = 2\text{Tr}[\Phi_n(\rho_2) - \Phi(\rho_2)]^+ \leq 2\text{Tr}\rho_{22} \longrightarrow 0, (n \longrightarrow \infty),$$

since $\text{Tr}\Phi_n(\rho_2) = \text{Tr}\Phi(\rho_2)$ implies

$$\begin{aligned} \|\Phi_n(\rho_2) - \Phi(\rho_2)\|_1 &= \text{Tr}(|\Phi_n(\rho_2) - \Phi(\rho_2)|) \\ &= \text{Tr}[\Phi_n(\rho_2) - \Phi(\rho_2)]^+ + \text{Tr}[\Phi_n(\rho_2) - \Phi(\rho_2)]^- \\ &= 2\text{Tr}[\Phi_n(\rho_2) - \Phi(\rho_2)]^+, \end{aligned}$$

and $\Phi_n(\rho_2) - \Phi(\rho_2) \leq (I - P_n) \rho_2 (I - P_n)$ yields $[\Phi_n(\rho_2) - \Phi(\rho_2)]^+ \leq P_+ (I - P_n) \rho_2 (I - P_n) P_+$, so

$$\begin{aligned} \text{Tr}[\Phi_n(\rho_2) - \Phi(\rho_2)]^+ &\leq \text{Tr}[P_+ (I - P_n) \rho_2 (I - P_n) P_+] \\ &\leq \text{Tr}[(I - P_n) \rho_2 (I - P_n)] \\ &= \text{Tr}\rho_{22}, \end{aligned}$$

where A^+ , A^- are the positive and negative parts of the self-adjoint operator A , and P_+ is an orthogonal projection on the range of $[\Phi_n(\rho_2) - \Phi(\rho_2)]^+$. By the spectral decomposition of ρ_1 , we

conclude that there exists an orthonormal basis $\{f_i\}_{i=1}^\infty$ of \mathcal{H} such that $\rho_1 = \sum_{i=1}^\infty \lambda_i(\rho_1) f_i \otimes f_i$. Define a unitary operator V by $Vf_i = e_i$, for $i = 1, 2, \dots$, so $\rho_1 = V^* \Phi(\rho_2) V$. Denote $\Psi_n(\rho) = V^* \Phi_n(\rho) V$ and $\Psi(\rho) = V^* \Phi(\rho) V$, for all $\rho \in S(\mathcal{H})$; then by the proof above, we similarly get $\lim_{n \rightarrow \infty} \|\Psi_n(\rho) - \Psi(\rho)\|_1 = \lim_{n \rightarrow \infty} \|\Phi_n(\rho) - \Phi(\rho)\|_1 = 0$, for all $\rho \in S(\mathcal{H})$. (b) \implies (c) is clear.

(c) \implies (a). Let $\rho_1 = \Psi(\rho_2) = \sum_{i=1}^\infty A_i \rho_2 A_i^*$. Applying Weyl's eigenvalue theorem, we get for $n = 1, 2, \dots$,

$$\sum_{i=1}^n \lambda_i(\rho_1) = \max_{P \in \mathcal{P}_n} \{Tr(\sum_{i=1}^\infty A_i \rho_2 A_i^* P)\} = \max_{P \in \mathcal{P}_n} \{Tr(\rho_2 \sum_{i=1}^\infty A_i^* P A_i)\}. \quad (3.13)$$

As $\sum_{i=1}^\infty A_i^* A_i = I$ and $\sum_{i=1}^\infty A_i A_i^* = I$, then $\sum_{i=1}^\infty A_i^* P A_i \leq I$ and

$$Tr(\sum_{i=1}^\infty A_i^* P A_i) = Tr(\sum_{i=1}^\infty A_i A_i^* P) = n.$$

For convenience, we denote $B = \sum_{i=1}^\infty A_i^* P A_i$ and $u_i = \langle B x_i, x_i \rangle$, where $\{x_i\}_{i=1}^\infty$ is an orthonormal basis of \mathcal{H} satisfying $\rho_2 = \sum_{i=1}^\infty \lambda_i(\rho_2) x_i \otimes x_i$. Thus $0 \leq u_i \leq 1$, and $\sum_{i=1}^\infty u_i = n$, so

$$\begin{aligned} Tr(\rho_2 B) = \sum_{i=1}^\infty \lambda_i(\rho_2) u_i &= \sum_{i=1}^n \lambda_i(\rho_2) u_i + \sum_{i=n+1}^\infty \lambda_i(\rho_2) u_i \\ &\leq \sum_{i=1}^n \lambda_i(\rho_2) u_i + \lambda_{n+1}(\rho_2) \sum_{i=n+1}^\infty u_i \\ &\leq \sum_{i=1}^n \lambda_i(\rho_2) u_i + \lambda_{n+1}(\rho_2) \sum_{i=1}^n (1 - u_i) \\ &\leq \sum_{i=1}^n \lambda_i(\rho_2). \end{aligned} \quad (3.14)$$

By equation (3.13) and inequality (3.14), we conclude that for all $n = 1, 2, \dots$, $\sum_{i=1}^n \lambda_i(\rho_1) \leq \sum_{i=1}^n \lambda_i(\rho_2)$ as desired.

Remark 3.4. In a finite dimensional Hilbert space, $\rho_1 \prec \rho_2$ is equivalent to $\rho_1 = \Phi(\rho_2)$, for some mixed unitary operation Φ . However, for an infinite dimensional Hilbert space, $\rho_1 \prec \rho_2$ does not imply $\rho_1 = \sum_{i=1}^\infty t_i U_i \rho_2 U_i^*$, where $\sum_{i=1}^\infty t_i = 1$, $t_i \geq 0$ and U_i are unitary operators for all i . Indeed, the condition $\rho_1 = \sum_{i=1}^\infty t_i U_i \rho_2 U_i^*$ yields $\dim(\ker(\rho_1)) \leq \dim(\ker(\rho_2))$. But we can supplement many zeros for $\lambda(\rho_1)$.

The following proposition was obtained in [10,17] for the finite case, in which the condition of injectivity of ρ may be dropped.

Proposition 3.5. Let $\rho \in S(\mathcal{H})$ and Φ be a bi-stochastic quantum operation. If ρ is injective and $S(\rho) = S(\Phi(\rho)) < \infty$, then $\Phi(\rho) = U \rho U^*$ for a unitary operator U .

Proof. Suppose $\Phi(\rho) = \sum_{i=1}^\infty A_i \rho A_i^*$. Then by Theorem 3.3, we have $\Phi(\rho) \prec \rho$. We claim that if ρ is injective, then so is $\Phi(\rho)$. We assume that $\Phi(\rho)$ were not injective, then there is a vector $x \neq 0$ that satisfies $\Phi(\rho)x = 0$, so $\langle \sum_{i=1}^\infty A_i \rho A_i^* x, x \rangle = 0$, which yields $A_i^* x = 0$, for all i . Thus $x = \sum_{i=1}^\infty A_i A_i^* x = 0$, which is a contradiction. It follows from Proposition 2.4 that $\lambda(\rho)^\downarrow = \lambda(\Phi(\rho))^\downarrow$, so by the spectral decomposition theorem, we get $\Phi(\rho) = U \rho U^*$ for a unitary operator U .

Remark 3.6. In an infinite dimensional Hilbert space, the condition that ρ is injective may not be dropped.

In the following, we shall characterize the structure of a quantum channel that does not change the von Neumann entropy of any quantum state. In [11], Molnár and Szokol gave the structure of the

map which preserves the relative entropy in a finite-dimensional Hilbert space. However, we cannot find the paper which study the structure of the map which preserves the von Neumann entropy. The following two lemmas are needed. As usual, $\mathcal{K}(\mathcal{H})$ is the set of all compact operators on \mathcal{H} .

Lemma 3.7. (see [14]) (i) For all $\rho \in S(\mathcal{H})$, $S(\rho) \geq 0$ and $S(\rho) = 0$ if and only if ρ is a pure state.

(ii) In a n dimension Hilbert space, $S(\rho) \leq \log n$, and $S(\rho) = \log n$ if and only if $\rho = \frac{I}{n}$.

Lemma 3.8. (see [9]) Let Φ be a quantum operation with $\Phi^\dagger(I) \leq I$. Then $\{B \in \mathcal{K}(\mathcal{H}) : \Phi(B) = B\} = \{B \in \mathcal{K}(\mathcal{H}) : \Phi^\dagger(B) = B\} \subseteq \mathcal{A}'$, where \mathcal{A}' is the set of commutators of $\mathcal{A} = \{A_i, A_i^* : A_i \in \mathcal{B}(\mathcal{H}), i = 1, 2, \dots\}$.

Theorem 3.9. Let Φ be a quantum channel on $\mathcal{B}(\mathcal{H})$. Then $S(\Phi(\rho)) = S(\rho)$ for all quantum states $\rho \in S(\mathcal{H})$ if and only if there exists an isometry $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\Phi(X) = V X V^*$ for all $X \in \mathcal{B}(\mathcal{H})$.

Proof. Sufficiency is clear.

Necessity. For any unit vector x , we have $S(\Phi(x \otimes x)) = S(x \otimes x) = 0$, so $\Phi(x \otimes x)$ is a rank one orthogonal projection. Thus there exists a unit vector z such that $\Phi(x \otimes x) = z \otimes z$.

Let $y \perp x$ be another vector of \mathcal{H} . For convenience, denote

$$\Phi(x \otimes x) = x' \otimes x' \text{ and } \Phi(y \otimes y) = y' \otimes y'.$$

Setting $\rho_0 = \frac{x \otimes x + y \otimes y}{2}$, we have

$$S\left(\frac{x' \otimes x' + y' \otimes y'}{2}\right) = S(\Phi(\rho_0)) = S(\rho_0) = 1. \quad (3.15)$$

In the following, we shall show $x' \perp y'$. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ denote the two-dimensional space spanned by x' and y' . Then $\Phi(\rho_0)$ can be treated as an operator from \mathcal{H}_0 into \mathcal{H}_0 . Suppose $y' = \alpha x' + \sqrt{1 - |\alpha|^2} x'^\perp$, where $0 \leq |\alpha| \leq 1$ and x'^\perp is a unit vector in \mathcal{H}_0 orthogonal to x' . Then we have the following matrix forms for the operators $x' \otimes x'$ and $y' \otimes y'$, respectively:

$$x' \otimes x' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad y' \otimes y' = \begin{pmatrix} |\alpha|^2 & \alpha \sqrt{1 - |\alpha|^2} \\ \bar{\alpha} \sqrt{1 - |\alpha|^2} & 1 - |\alpha|^2 \end{pmatrix},$$

where $\bar{\alpha}$ is the complex conjugate of α . Then

$$\frac{x' \otimes x' + y' \otimes y'}{2} = \begin{pmatrix} \frac{1 + |\alpha|^2}{2} & \frac{\alpha \sqrt{1 - |\alpha|^2}}{2} \\ \frac{\bar{\alpha} \sqrt{1 - |\alpha|^2}}{2} & \frac{1 - |\alpha|^2}{2} \end{pmatrix}. \quad (3.16)$$

By a direct calculation, we conclude that the characteristic polynomial of (3.16) is

$$\lambda^2 - \lambda + \frac{1 - |\alpha|^2}{4} = 0. \quad (3.17)$$

Furthermore, Lemma 3.7 (ii) and equation (3.15) imply that equation (3.17) has two equal roots $\lambda_1 = \lambda_2 = \frac{1}{2}$, which yields $\alpha = 0$. Thus, $x' \perp y'$ as required. From the proof above, we know that the map Φ sends orthogonal pure states to orthogonal pure states. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of \mathcal{H} and P_n be the orthogonal projection onto the subspace spanned by $\{e_i\}_{i=1}^n$. Then

$$\Phi(I) = \lim_{n \rightarrow \infty} \Phi(P_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Phi(e_i \otimes e_i),$$

so $\Phi(I)$ is an infinite dimensional orthogonal projection.

Let $\rho \in S(\mathcal{H})$ be injective and $\rho = \sum_{i=1}^\infty \lambda_i x_i \otimes x_i$, where $\{x_i\}_{i=1}^\infty$ is an orthonormal basis and λ_i are the eigenvalues of ρ . Then

$$\Phi(\rho) = \lim_{n \rightarrow \infty} \Phi\left(\sum_{i=1}^n \lambda_i x_i \otimes x_i\right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i \Phi(x_i \otimes x_i) = \sum_{i=1}^\infty \lambda_i x'_i \otimes x'_i.$$

Denote $V_\rho x_i = x'_i$ for all i , then $\Phi(\rho) = V_\rho \rho V_\rho^*$.

By the Kraus Theorem, we have $\Phi(\rho) = \sum_{i=1}^\infty A_i \rho A_i^*$, where $A_i \in \mathcal{B}(\mathcal{H})$, and $\sum_{i=1}^\infty A_i^* A_i = I$, as Φ is trace-preserving. Then

$$\sum_{i=1}^\infty A_i \rho A_i^* = V_\rho \rho V_\rho^*,$$

which yields

$$\sum_{i=1}^\infty V_\rho^* A_i \rho A_i^* V_\rho = \rho, \quad (3.18)$$

for all injective $\rho \in S(\mathcal{H})$. Furthermore, $\sum_{i=1}^\infty A_i A_i^* = \Phi(I) \leq I$, so

$$\sum_{i=1}^\infty V_\rho^* A_i A_i^* V_\rho \leq I \text{ and } \sum_{i=1}^\infty A_i^* V_\rho V_\rho^* A_i \leq \sum_{i=1}^\infty A_i^* A_i = I. \quad (3.19)$$

Then Lemma 3.8 and equation (3.18) imply

$$\rho \sum_{i=1}^\infty V_\rho^* A_i A_i^* V_\rho = \sum_{i=1}^\infty V_\rho^* A_i \rho A_i^* V_\rho = \rho,$$

and

$$\rho \sum_{i=1}^\infty A_i^* V_\rho V_\rho^* A_i = \sum_{i=1}^\infty A_i^* V_\rho \rho V_\rho^* A_i = \rho, \quad (3.20)$$

so

$$I = \sum_{i=1}^\infty A_i^* V_\rho V_\rho^* A_i, \quad (3.21)$$

which yields

$$\sum_{i=1}^{\infty} A_i^* V_{\rho} V_{\rho}^* A_i = I = \sum_{i=1}^{\infty} A_i^* A_i. \quad (3.22)$$

Thus

$$\sum_{i=1}^{\infty} A_i^* (I - V_{\rho} V_{\rho}^*) A_i = 0,$$

which implies $A_i^* (I - V_{\rho} V_{\rho}^*) = 0$, that is

$$A_i^* = A_i^* V_{\rho} V_{\rho}^*, \quad \text{for all injective } \rho \in S(\mathcal{H}). \quad (3.23)$$

Using equations (3.18) and (3.20), we get

$$\sum_i A_i^* V_{\rho} \left(\sum_j V_{\rho}^* A_j \rho A_j^* V_{\rho} \right) V_{\rho}^* A_i = \rho, \quad (3.24)$$

so equation (3.23) implies

$$\sum_{i,j} A_i^* A_j \rho A_j^* A_i = \rho, \quad (3.25)$$

which is equivalent to $\Phi^{\dagger} \circ \Phi(\rho) = \rho$, for all injective $\rho \in S(\mathcal{H})$.

For any unit vector x , we claim that there exists a sequence of injective states $\rho_n \in S(\mathcal{H})$ such that $\rho_n \xrightarrow{W^*} x \otimes x$. Indeed, let $\{f_i\}_{i=1}^{\infty}$ be an orthonormal basis of the orthogonal complement subspace of x , and $\rho_n = (1 - \frac{1}{n})x \otimes x + \frac{1}{n} \sum_{i=1}^{\infty} (\frac{1}{2^i} f_i \otimes f_i)$. Then $\lim_{n \rightarrow \infty} \text{Tr}(|\rho_n - x \otimes x|) = 0$, so $\lim_{n \rightarrow \infty} \text{Tr}[(\rho_n - x \otimes x)X] = 0$, for all $X \in \mathcal{B}(\mathcal{H})$. Thus

$$x \otimes x = \lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} \Phi^{\dagger} \circ \Phi(\rho_n) = \Phi^{\dagger} \circ \Phi(x \otimes x),$$

which says

$$\sum_{i,j} A_i^* A_j (x \otimes x) A_j^* A_i = \Phi^{\dagger} \circ \Phi(x \otimes x) = x \otimes x$$

for all rank one projections $x \otimes x$. Then by Lemma 3.8 again, we have $A_i^* A_j (x \otimes x) = (x \otimes x) A_i^* A_j$, which implies that for all $i, j = 1, 2, \dots$, $A_j^* A_i = \lambda_{ji} I$ and $A_i^* A_i = \lambda_{ii} I$, so $\lambda_{ii} > 0$ and $\sum_i \lambda_{ii} = 1$. It is clear that

$$\begin{pmatrix} A_1^* \\ A_2^* \\ \vdots \\ A_n^* \\ \vdots \end{pmatrix} \begin{pmatrix} A_1 & A_2 & \cdots & A_n & \cdots \end{pmatrix} = \begin{pmatrix} A_1^* A_1 & A_1^* A_2 & \cdots & A_1^* A_n & \cdots \\ A_2^* A_1 & A_2^* A_2 & \cdots & A_2^* A_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n^* A_1 & A_n^* A_2 & \cdots & A_n^* A_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \\ = \begin{pmatrix} \lambda_{11} I & \lambda_{12} I & \cdots & \lambda_{1n} I & \cdots \\ \lambda_{21} I & \lambda_{22} I & \cdots & \lambda_{2n} I & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1} I & \lambda_{n2} I & \cdots & \lambda_{nn} I & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Denote

$$M = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1n} & \dots \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{n1} & \lambda_{n2} & \dots & \lambda_{nn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

so $\lambda_{ji} = \overline{\lambda_{ij}}$ and

$$|\lambda_{ij}|^2 \leq \lambda_{ii}\lambda_{jj}, \text{ for } 1 \leq i, j \quad (3.26)$$

since $M \geq 0$. Further equation (3.25) implies $\sum_{i,j} A_j^* A_i A_i^* A_j = I$, which yields

$$\sum_{i,j} |\lambda_{ij}|^2 = 1 = \left(\sum_{i=1}^{\infty} \lambda_{ii} \right)^2.$$

Then by a direct calculation, we get

$$\sum_{i \neq j} |\lambda_{ij}|^2 = \sum_{i \neq j} \lambda_{ii} \lambda_{jj},$$

so equation (3.26) implies

$$|\lambda_{ij}|^2 = \lambda_{ii} \lambda_{jj}, \text{ for } 1 \leq i, j \leq \infty.$$

Thus

$$A_j^* A_1 = \lambda_{j1} I = \sqrt{\lambda_{11} \lambda_{jj}} e^{\sqrt{-1}\theta_j} I,$$

for $j = 1, 2, \dots$. We denote $V_j = \frac{A_j}{\sqrt{\lambda_{jj}}}$, so V_j are isometric operators for $j = 1, 2, \dots$, which yields $V_j^* V_1 = e^{\sqrt{-1}\theta_j} I$. Then $V_j^* V_1 V_1^* = e^{\sqrt{-1}\theta_j} V_j^*$ and $V_j V_j^* V_1 = e^{\sqrt{-1}\theta_j} V_j$. Hence

$$V_1 V_1^* V_j V_j^* V_1 V_1^* = V_1 V_1^* \text{ and } V_j V_j^* V_1 V_1^* V_j V_j^* = V_j V_j^*,$$

so we get

$$V_1 V_1^* (I - V_j V_j^*) = 0 \text{ and } V_j V_j^* (I - V_1 V_1^*) = 0,$$

then $V_1 V_1^* = V_j V_j^*$, for $j = 1, 2, \dots$. Thus for all $X \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} A_j X A_j^* &= V_1 V_1^* A_j X A_j^* V_1 V_1^* \\ &= V_1 \frac{A_1^* A_j X A_j^* A_1}{\lambda_{11}} V_1^* \\ &= \lambda_{jj} V_1 X V_1^*, \end{aligned}$$

which implies

$$\Phi(X) = A_1 X A_1^* + A_2 X A_2^* + \dots + A_n X A_n^* + \dots = V_1 X V_1^*.$$

The following result is clear, by Theorem 3.9.

Corollary 3.10. Let Φ be a bi-stochastic quantum operation. Then $S(\Phi(\rho)) = S(\rho)$ for all quantum state $\rho \in S(\mathcal{H})$ if and only if there exists a unitary matrix U such that $\Phi(\rho) = U \rho U^\dagger$.

References

- [1] P.M. Alberti and A. Uhlmann. Stochasticity and partial order: doubly stochastic maps and unitary mixing. Dordrecht, Boston, 1982.
- [2] J. Antezana, P. Massey, M. Ruiz, and D. Stojanoff, The Schur-Horn Theorem for operators and frames with prescribed norms and frame operator, Illinois J of Math., 51 (2007) 537-560.
- [3] W. Arveson and R. V. Kadison, Diagonals of self-adjoint operators. Operator theory, operator algebras, and applications, Contemp. Math., 414 (2006) 247-263.
- [4] G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities. 2nd Ed. Cambridge University Press, 1973.
- [5] F. Hiai, M. Mosonyi, D. Petz, C. Bény, Quantum f-divergences and error correction, Rev. Math. Phys. 23 (2011) 691-747.
- [6] I. Gohberg and A. Markus, Some relations between eigenvalues and matrix elements of linear operators Mat. Sb. 64 (106) (1964), 481-496 (Russian); Amer. Math. Soc. Transl. (2) 52 (1966) 201-216 (English)
- [7] V. Kaftal and G. Weiss, An infinite dimensional Schur-Horn theorem and majorization theory, J. Functional Analysis, 259 (2010) 3115-3162.
- [8] K. Kraus, General state changes in quantum theory, Ann. Physics 64 (1971), 311-335.
- [9] Y. Li, Fixed points of dual quantum operations, Journal of Mathematical Analysis and Applications, 382 (2011) 172-179.
- [10] Y. Li, Y. Wang, Further results on entropy and separability, Journal of Physics A: Mathematical and Theoretical, 45 (2012) 385305.
- [11] L. Molnár, P. Szokol, Maps on states preserving the relative entropy II, Linear Algebra Appl. 432 (2010) 3343-3350.
- [12] A. Neumann, An infinite-dimensional generalization of the Schur-Horn convexity theorem. J. Funct. Anal., 161 (1999) 418-451.
- [13] M.A. Nielsen. An introduction of Majorization and its Applications to Quantum Mechanics. Queensland 4072, Australia (2002).
- [14] M.A. Nielsen, I.L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, Cambridge (2000).
- [15] D. Petz, Quantum Information Theory and Quantum Statistics, Springer, Berlin, Heidelberg, 2008.

- [16] H. Weyl. Inequalities between the two kinds of eigenvalues of a linear transformation. Proc. N.A.S. (USA), 35 (1949) 408-411.
- [17] L. Zhang, J.D. Wu, Von Neumann entropy-preserving quantum operation, Phys. Lett. A 375 (2011) 4163-4165.